A Preconditioner Construction for Large Scale 3D Magnetostatic Problems

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***Abstract* —An iterative domain decomposition method is applied to numerical analysis of 3-Dimensional (3D) nonlinear magnetostatic problems taking the magnetic vector potential as an unknown function. The iterative domain decomposition method is combined with the Preconditioned Conjugate Gradient (PCG) procedure and the Hierarchical Domain Decomposition Method (HDDM) which is adopted in parallel computing. Our previously employed preconditioner was the Neumann-Neumann one. Numerical results showed that the preconditioner was only effective for smaller problems. In this paper, we consider its improvement with the Balancing Domain Decomposition (BDD) like preconditioner.**

I. INTRODUCTION

Among various parallel computing techniques, the Domain Decomposition Method (DDM) is a promising one in solving 3-Dimensional (3D) nonlinear magnetostatic problems with the magnetic vector potential *A* as an unknown function. In general DDM decomposes the whole domain into independent subdomains. Different methods can be chosen for solving the equations employed on these subdomains. On the other hand, DDM needs an iteration process in solving the interface problem to obtain the final solution. The nonlinear simultaneous equations are solved with the Newton iteration [1]. In this work, for simplicity, we focus our attention on the linear equation solving at each nonlinear iteration. We have also adopted a perturbation problem to explain conveniently the problem, for simplicity. Owing to this technique, a direct method can be used to solve the matrix equations on subdomains and a Preconditioned Conjugated Gradient (PCG) method can be used for the interface problem. In the overall parallel computing of the linear equation, the Hierarchical Domain Decomposition Method (HDDM) [2] is used.

The performance of the perturbation problem with the direct solver was successfully compared with the InComplete Conjugate Gradient (ICCG) method to solve subdomain problems [3]. Our employed CG preconditioner was the Neumann-Neumann one. For small number of subdomains, this preconditioner was suitable. But due to the absence of a coarse problem, the convergence with the preconditioner decayed rapidly for large number of subdomains. An investigation of the preconditioner for large scale 3D magnetostatic problems with large number of subdomains produced no effective result so far. This is an important problem that we are currently working on.

This extended abstract is arranged as follows. Iterative domain decomposition method is described in Section 2. The interface problem with the Neumann-Neumann preconditioner and the Balancing Domain Decomposition (BDD) like preconditioner is discussed in Section 3.

II. ITERATIVE DOMAIN DECOMPOSITION METHOD

We consider 3D nonlinear magnetostatic problems using the *A* method and the Newton method, see [1]. Then, we introduce an iterative domain decomposition method to this method. Let us denote the linear finite element equation of the *A* method by the matrix form as follows:

$$Ku = f \tag{1}$$

where K denotes the coefficient matrix, u the unknown vector, and f the known vector.

The polyhedral domain Ω is partitioned into the non-overlapping subdomains. Then the linear system (1) is rewritten as follows:

$$\begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix} \begin{pmatrix} u_I \\ u_B \end{pmatrix} = \begin{pmatrix} f_I \\ f_B \end{pmatrix} \tag{2}$$

where the subscripts I and B correspond to the nodal points in the interior of subdomains and on the interface boundary.

At first the unknown vector u_B is obtained from the algorithm based on the CG method to the interface problem, see [1]. After solving u_B, the unknown u_I is obtained from:

$$K_{II}u_I = f_I - K_{IB}u_B . \tag{3}$$

The vector u_I can be solved independently in each subdomain. Hence we can get the unknown u in the whole domain.

In the previous studies [1], the vector u_I was solved by the ICCG method, because the finite element equation of the *A* method that neglects the Lagrange multiplier p is singular. Since the perturbation problem is considered in this paper, for simplicity, the direct method is used to solve the equations in subdomains.

III. PRECONDITIONERS

We then consider a non-overlapping partition of the domain Ω, consisting of subdomians, also called substructures $\left\{\Omega^{(i)}\right\}_{i=1,\dots N}$. We also define the interface as

$$\Gamma \equiv \bigcup_{i=1}^{N} \partial\Omega^{(i)} \setminus \Gamma_E , \tag{4}$$

where N is the number of subdomains and Γ_E is the essential boundary of the domain Ω. For the given magnetic reluctivity, the finite element discretization gives a symmetric positive definite linear system in the perturbation problem. The Degrees Of Freedom (DOF) inside subdomains are eliminated in parallel by using any direct method. We are then left with a linear system involving only DOF on Γ. If a local vector in $\Omega^{(i)}$ is divided into two subvectors; DOF corresponding to edges inside $\Omega^{(i)}$ and on $\partial\Omega^{(i)} \setminus \Gamma_E$, respectively, the local stiffness matrix of $K^{(i)}$ can be written as

$$K^{(i)} = \begin{pmatrix} K_{II}^{(i)} & K_{IB}^{(i)} \\ K_{IB}^{(i)T} & K_{BB}^{(i)} \end{pmatrix}. \quad (5)$$

Let $W^{(i)}$ be the space of interface DOF for the subdomain $\Omega^{(i)}$ and W be the space of all DOF on Γ. After eliminating DOF inside subdomains, the original problem reduces to a problem with smaller dimension;

$$Su_B = g, \qquad u_B \in W, \quad (6)$$

where $S = \sum_{i=1}^{N} R_B^{(i)} S^{(i)} R_B^{(i)T}$ is the global Schur complement matrix related to Γ and g is the resultant right hand side vector. We define the operators:

$$S : W \to W, \quad S^{(i)} : W^{(i)} \to W^{(i)}, \quad R_B^{(i)} : W^{(i)} \to W.$$

$R_B^{(i)T}$ is the transpose of $R_B^{(i)}$. The local Schur complement $S^{(i)}$ is defined as

$$S^{(i)} \equiv K_{BB}^{(i)} - K_{IB}^{(i)T}\left(K_{II}^{(i)}\right)^{-1} K_{IB}^{(i)}. \quad (7)$$

The problem (6) is solved by a PCG method which requires to solve the following auxiliary problem:

$$z = M^{-1} r \quad (8)$$

where *r* is the residual of (6) and *M* is a preconditioner. In the previous research [3], we tried to implement the Neumann-Neumann preconditioner without a coarse problem. Due to the absence of the coarse problem, it was restricted for problems with small number of subdomains. A BDD like preconditioner, that is, the Neumann-Neumann one with a coarse problem is the present challenge of this research.

The Neumann-Neumann preconditiner can be rewritten as

$$M_{NN} = \sum_{i=1}^{N} R_B^{(i)} D^{(i)} S^{(i)\dagger} D^{(i)T} R_B^{(i)T}, \quad (9)$$

where $S^{(i)\dagger}$ denotes the generalized inverse of $S^{(i)}$ and $D^{(i)}$ is the weight matrix [4]. Then, we introduce the Balancing Domain Decomposition (BDD) preconditioner of Mandel [4] as follows:

$$M_{BDD}^{-1} = \left((I-P)M_{NN}S(I-P)+P\right)S^{-1}, \quad (10)$$

where *P* is the S-orthogonal projection onto a subspace U of *W* defined by

$$U = \left\{ u \in W,\ u = \sum_{i=1}^{N} R_B^{(i)} D^{(i)} u^{(i)},\ u^{(i)} \in Range\ Z^{(i)} \right\}. \quad (11)$$

Here, $Z^{(i)}$ is $\dim W^{(i)} \times m^{(i)}$ matrices of full column rank $(0 \le m^{(i)} \le \dim W^{(i)})$ such that

$$Null\ S^{(i)} \subset Range\ Z^{(i)} \qquad i = 1, \cdots\cdots, N. \quad (12)$$

Setting of a suitable $Z^{(i)}$ is a crucial point in the BDD approach.

In this work, we formally follow the above BDD algorithm for the original problem (not for the perturbed problem) and construct a BDD like preconditioner. Then our construction of $Z^{(i)}$ is as follows:

Step 1: Select one nodal point P_j (a midpoint on one side of a tetrahedral element) on which the interface DOF is defined.

Step 2: Denote two vertexes whose midpoint is P_j as P_{j+1} and P_{j-1}.

Step 3: Construct the row vector of $Z^{(i)}$ corresponding to the nodal point P_j , whose DOF are edge components of the Nedelec element, as grad φ_{j+1} + grad φ_{j-1}, where φ_{j+1} or φ_{j-1} is a piecewise linear basis function with respect to a vertex P_{j+1} or P_{j-1}, respectively.

IV. CONCLUDING REMARKS

Using a purturbation technique, an iterative domain decomposition method was previously applied to 3D nonlinear magnetostatic problems [3]. In the present research, it is very important for us to reduce number of iterations and computation time. As one possibility, we are trying to implement the Neumann-Neumann preconditioner with a coarse problem which has been successfully used in structural analysis, thermal analysis and incompressible viscous flow analysis, see [5] for example. For magnetic field problems, the similar approach in this paper is expected to be effective. Numerical results will be shown in the conference.

V. REFERENCES

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